كلية الحاسبات والذكاء الإصطناعي

BS111

Probability and Statistics

Lecture 04

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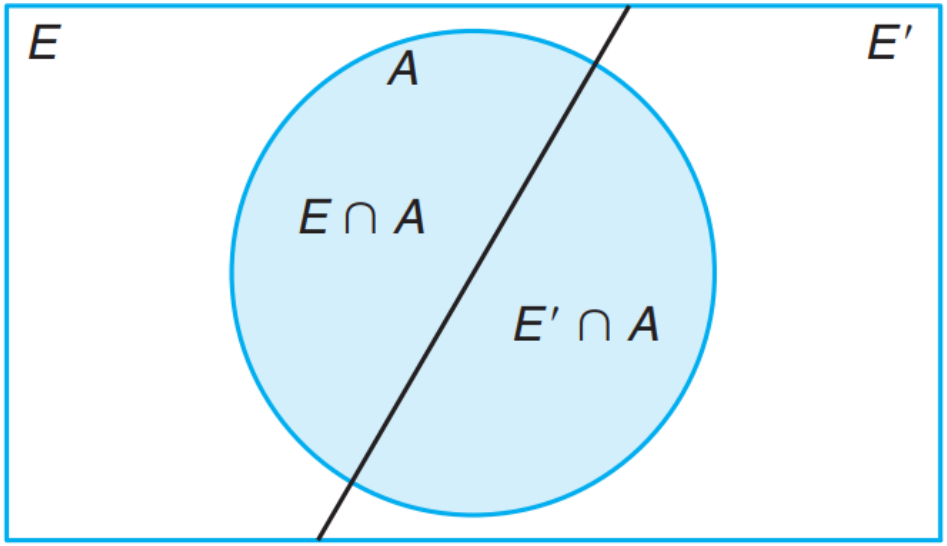
Chapter 1: Probability

- Sample Space.
- Events.
- Counting Techniques.
- Probability of an Event.
- Additive Rules.
- Conditional Probability.
- Independence, and the Product Rule.
- Total Probability Rule.
- Bayes' Rule.

Total Probability Rule (1/10)

Total Probability Rule:



$$\begin{aligned} \boldsymbol{P(A)} &= \boldsymbol{P(E \cap A) \cup P(E' \cap A)} \\ &= \boldsymbol{P(E \cap A) + P(E' \cap A)} \\ &= \boldsymbol{P(A|E)P(E) + P(A|E')P(E')} \end{aligned}$$

Total Probability Rule (2/10)

Example10:

Consider the information about contamination in the following table.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not high	0.8

$P(F|H) = 0.1$

$P(F|H') = 0.005$

$P(H) = 0.2$

$P(H') = 0.8$

$P(F)$?

Let F denote the event that the product fails, and let H denote the event that the chip is exposed to high levels of contamination.

Total Probability Rule (3/10)

Example10:

Consider the information about contamination in the following table.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not high	0.8

$P(F|H) = 0.1$

$P(F|H') = 0.005$

$P(H) = 0.2$

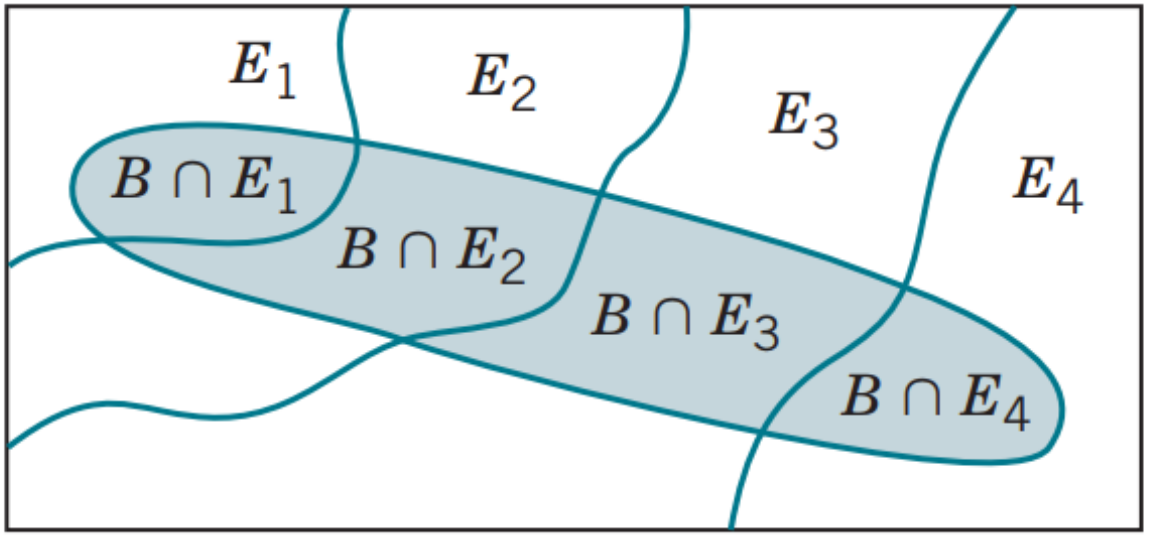
$P(H') = 0.8$

$P(F)$?

$$P(F) = P(F|H)P(H) + P(F|H')P(H')$$
$$= (0.1)(0.2) + (0.005)(0.8) = 0.024$$

Total Probability Rule (4/10)

Total Probability Rule (Multiple Events):



$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + P(B \cap E_3) + P(B \cap E_4)$$

Total Probability Rule (5/10)

Example11:

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Total Probability Rule (5/10)

Example11:

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

D: the product is defective. Find $P(D)$?

Total Probability Rule (6/10)

Example11: $P(B_1) = 0.3, P(B_2) = 0.45, P(B_3) = 0.25$

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

$$P(D|B_1) = 0.02,$$
$$P(D|B_2) = 0.03,$$
$$P(D|B_3) = 0.02.$$

Total Probability Rule (7/10)

Example11: $P(B_1) = 0.3, P(B_2) = 0.45, P(B_3) = 0.25$

$P(D|B_1) = 0.02,$
$P(D|B_2) = 0.03,$
$P(D|B_3) = 0.02.$

Applying the total probability rule , we can write

$$P(D)$$
$$= P(D|B_1)P(B_1) + P(D|B_2)P(B_2) + P(D|B_3)P(B_3)$$
$$= 0.02(0.3) + 0.03(0.45) + 0.02(0.25) = 0.0245$$

Total Probability Rule (8/10)

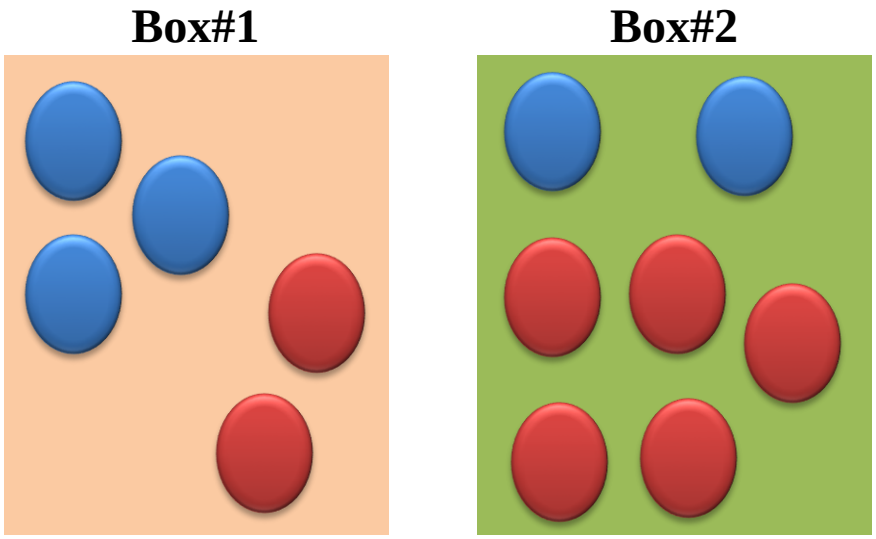
Example12:

Box#1 contains 2 red balls and 3 blue balls; Box#2 contains 5 red balls and 2 blue balls. If the selection of two boxes is equally likely, and you selected one ball, what is the probability that it is red?

Total Probability Rule (8/10)

Example12:

Box#1 contains 2 red balls and 3 blue balls; Box#2 contains 5 red balls and 2 blue balls. If the selection of two boxes is equally likely, and you selected one ball, what is the probability that it is red?



Total Probability Rule (9/10)

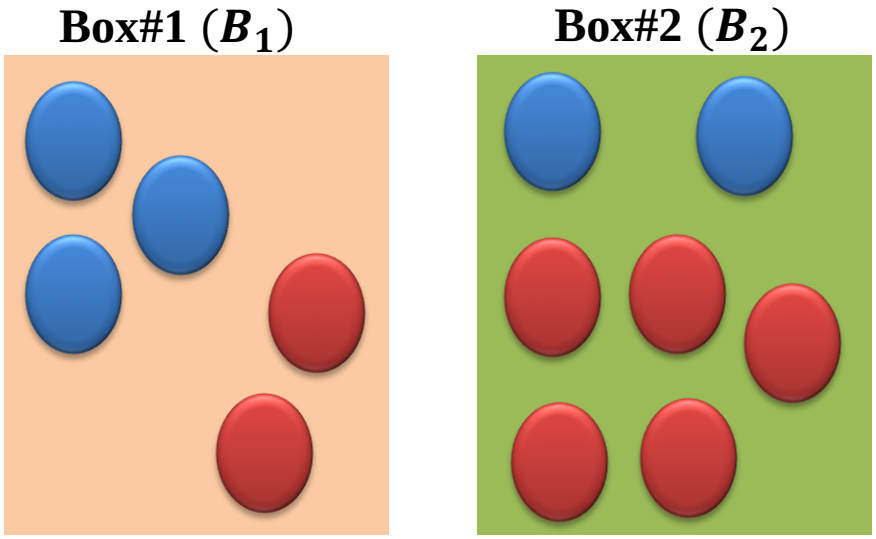
Example12:

Box#1 contains 2 red balls and 3 blue balls; Box#2 contains 5 red balls and 2 blue balls**. If the selection of two boxes is equally likely**, and you selected one ball, what is the probability that it is red?

$P(B_1) = P(B_2) = 0.5$

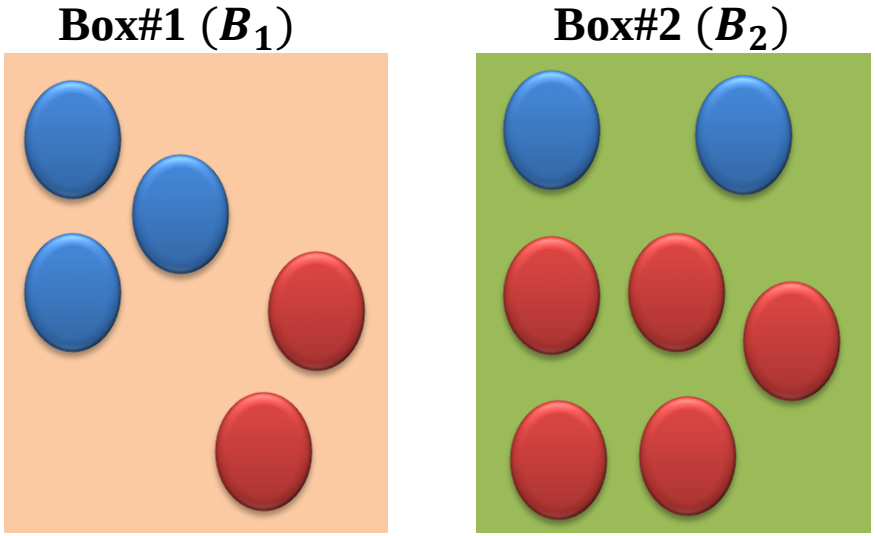
$R: read, \qquad B: blue$

Find $P(R)$?



Total Probability Rule (10/10)

Example12:



$$P(B_1) = P(B_2) = 0.5$$

$$R: read, \qquad B: blue$$

$$P(R|B_1) = {}^{2}/_{5} = 0.4$$

$$P(R|B_2) = {}^{5}/_{7} = 0.7143$$

$$P(R) = P(R|B_1)P(B_1) + P(R|B_2)P(B_2)$$

$$= (0.4)(0.5) + (0.7143)(0.5) = 0.55715$$

Bayes' Rule (1/11)

From the definition of conditional probability,

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Now, considering the second and last terms in the preceding expression, we can write

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \quad \text{for} \quad P(B) > 0$$

Bayes' Rule (2/11)

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1 \mid B) = \frac{P(B \mid E_1)P(E_1)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \cdots + P(B \mid E_k)P(E_k)}$$

for $P(B) > 0$

Bayes' Rule (3/11)

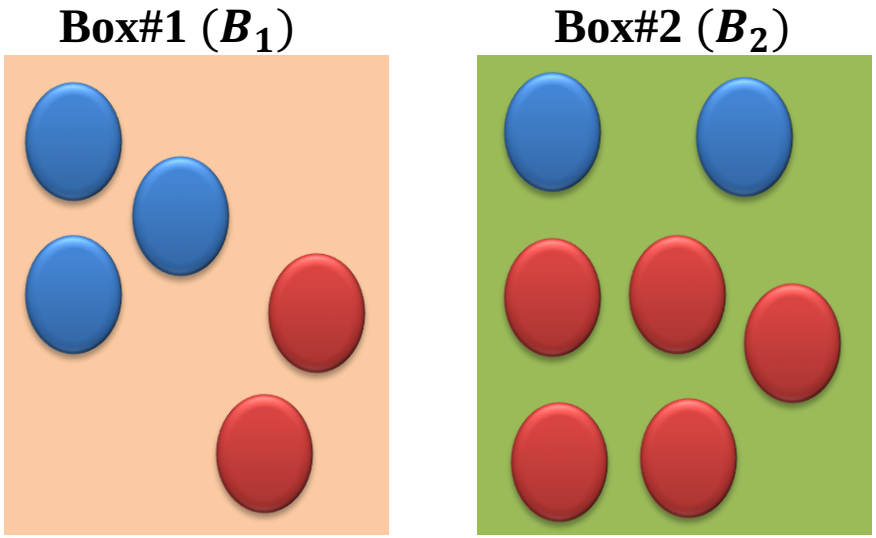
Example1:

Box#1 contains 2 red balls and 3 blue balls; Box#2 contains 5 red balls and 2 blue balls. If the selection of two boxes is equally likely, and the selected ball was red, what is the probability that it is from Box#1?

$$P(B_1) = P(B_2) = 0.5$$

$$R: read, \qquad B: blue$$

Find $P(B_1|R)$?



Bayes' Rule (4/11)

Example1:

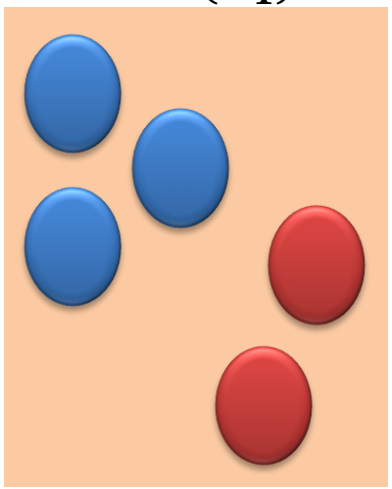
$$P(B_1) = P(B_2) = 0.5$$

$$R: read, \qquad B: blue$$

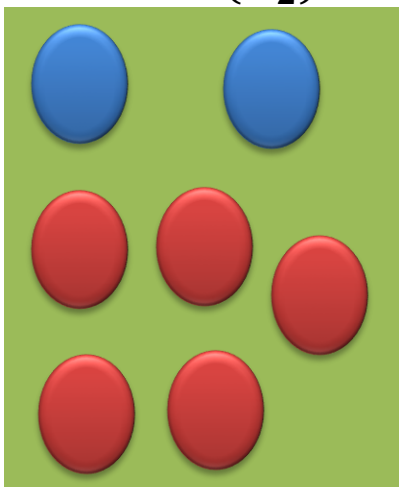
$$P(R|B_1) = {}^{2}/_{5} = 0.4$$

$$P(R|B_2) = {}^{5}/_{7} = 0.7143$$

Box#1 (B_1)



Box#2 (B_2)



Bayes' Rule (5/11)

Example1:

$$P(B_1) = P(B_2) = 0.5$$

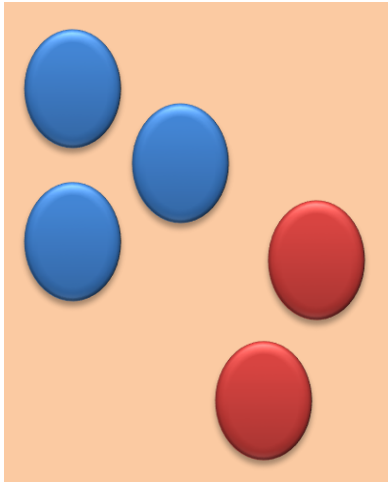
$$R: read, \qquad B: blue$$

$$P(R|B_1) = {}^{2}/_{5} = 0.4$$

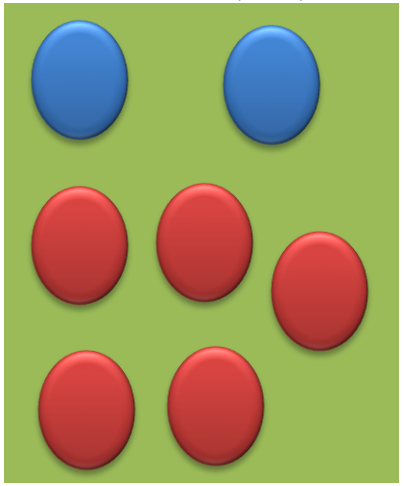
$$P(R|B_2) = {}^{5}/_{7} = 0.7143$$

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R)} = \frac{(0.4)(0.5)}{P(R)}$$

Box#1 (B_1)

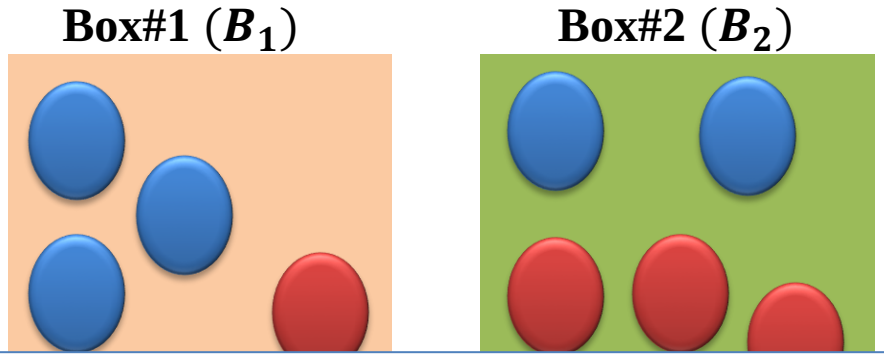


Box#2 (B_2)



Bayes' Rule (6/11)

Example1:



$$P(B_1) = P(B_2) = 0.5$$

$R: read, \qquad B: blue$

$$P(R|B_1) = {}^{2}/_{5}$$

$$P(R|B_2) = {}^{5}/_{7}$$

$$P(R) = P(R|B_1)P(B_1) + P(R|B_2)P(B_2)$$

$$= (0.4)(0.5) + (0.7143)(0.5) = 0.55715$$

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R)} = \frac{(0.4)(0.5)}{P(R)}$$

Bayes' Rule (7/11)

Example1:

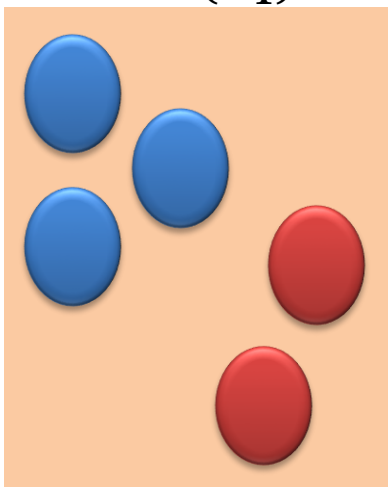
$$P(B_1) = P(B_2) = 0.5$$

$$R: read, \qquad B: blue$$

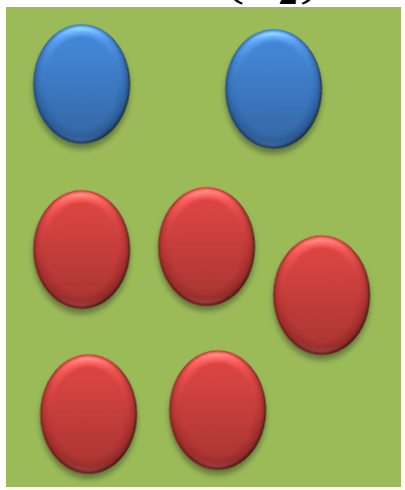
$$P(R|B_1) = {}^{2}/_{5} = 0.4$$

$$P(R|B_2) = {}^{5}/_{7} = 0.7143$$

Box#1 (B_1)



Box#2 (B_2)



$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R)} = \frac{(0.4)(0.5)}{P(R)} = \frac{0.2}{0.55715} = 0.35897$$

Bayes' Rule (8/11)

Example2:

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. If a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

D: the product is defective. Find $P(B_3|D)$?

Bayes' Rule (8/11)

Example2: $P(B_1) = 0.3, P(B_2) = 0.45, P(B_3) = 0.25$

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. If a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

$$P(D|B_1) = 0.02,$$
$$P(D|B_2) = 0.03,$$
$$P(D|B_3) = 0.02.$$

Bayes' Rule (9/11)

Example2: $P(B_1) = 0.3, P(B_2) = 0.45, P(B_3) = 0.25$

$$P(D|B_1) = 0.02,$$
$$P(D|B_2) = 0.03,$$
$$P(D|B_3) = 0.02.$$

Using Bayes' rule to write

$$P(B_3|D) = \frac{P(D|B_3)P(B_3)}{P(D)}$$

Bayes' Rule (10/11)

Example2:

Applying the total probability rule , we can write

$$P(D)$$
$$= P(D|B_1)P(B_1) + P(D|B_2)P(B_2) + P(D|B_3)P(B_3)$$
$$= 0.02(0.3) + 0.03(0.45) + 0.02(0.25) = 0.0245$$

Using Bayes' rule to write

$$P(B_3|D) = \frac{P(D|B_3)P(B_3)}{P(D)}$$

Bayes' Rule (11/11)

Example2: $P(B_1) = 0.3, P(B_2) = 0.45, P(B_3) = 0.25$

$P(D|B_1) = 0.02,$
$P(D|B_2) = 0.03,$
$P(D|B_3) = 0.02.$

Using Bayes' rule to write

$$P(B_3|D) = \frac{P(D|B_3)P(B_3)}{0.0245} = \frac{(0.02)(0.25)}{0.0245} = 0.2041$$

Chapter 2: Random Variable

- Discrete Random Variables.
- Probability Mass Functions.
- Cumulative Distribution Functions.
- Discrete R.V. (Mean and Variance).
- Continuous Random Variables.
- Probability Density Functions.
- Continuous R.V. (Mean and Variance).
- Joint Probability Distributions.

Discrete Random Variables (1/3)

Random Variable

- Is a function that assigns a real number to each outcome in the sample space of random experiment. Denoted by an uppercase letter such as X

A Discrete Random Variable

- Is a random variable with a finite (or countable infinite) range.
- The possible values of X may be listed as $x_1, x_2, \ldots$

Discrete Random Variables (2/3)

Example1

- Flipping a coin of two times. Let X is the number of heads.

Discrete Random Variables (3/3)

Example1

- Flipping a coin of two times. Let X is the number of heads.

Answer:

$$S = \{HH, HT, TH, TT\}$$

$$2 \quad 1 \quad 1 \quad 0$$

$$x = 0, 1, 2$$

$$P(0) = \frac{1}{4}, \qquad P(1) = \frac{2}{4}, \qquad P(2) = \frac{1}{4}$$

Probability Mass Fun. (1/14)

Probability Mass Function

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, a **probability mass function** is a function such that

(1) $f(x_i) \geq 0$

(2) $\sum_{i=1}^{n} f(x_i) = 1$

(3) $f(x_i) = P(X = x_i)$

x_i	x_1	x_2	x_3	x_4	x_5
$f(x_i) = P(x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	$P(x_4)$	$P(x_5)$

Probability Mass Fun. (2/14)

Example1

Verify that the function is a probability mass function:

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Probability Mass Fun. (3/14)

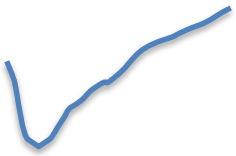
Example1

Verify that the function is a probability mass function:

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Answer:

$$\sum P(x_i) = 1, \qquad P(x_i) \geq 0$$



Probability Mass Fun. (4/14)

Example2

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Find:

a. $P(X \leq 2)$

b. $P(X > -2)$

c. $P(-1 \leq X \leq 1)$

d. $P(X \leq -1 \text{ or } X = 2)$

Probability Mass Fun. (5/14)

Example2

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Answer:

a. $P(X \leq 2) = 1$

b. $P(X > -2) = \frac{7}{8}$

c. $P(-1 \leq X \leq 1) = \frac{6}{8}$

d. $P(X \leq -1 \quad \text{or} \quad X = 2) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8}$

Probability Mass Fun. (6/14)

Example3

Two balls are drawn in succession without replacement from a box containing 4 red balls and 3 black balls. The possible outcomes and the values y of the random variable Y , where y is the number of red balls, are

Sample Space	y
RR	2
RB	1
BR	1
BB	0

Probability Mass Fun. (7/14)

Example3

Sample Space	y
RR	2
RB	1
BR	1
BB	0

No Red Balls

$$f(0) = P(Y = 0) = \frac{\binom{4}{0}\binom{3}{2}}{\binom{7}{2}} = \frac{3}{21} = \frac{1}{7}$$

Probability Mass Fun. (7/14)

Example3

Sample Space	y
RR	2
RB	1
BR	1
BB	0

One Red Ball

$$f(1) = P(Y = 1) = \frac{\binom{4}{1}\binom{3}{1}}{\binom{7}{2}} = \frac{12}{21} = \frac{4}{7}$$

Probability Mass Fun. (7/14)

Example3

Sample Space	y
RR	2
RB	1
BR	1
BB	0

Two Red Balls

$$f(2) = P(Y = 2) = \frac{\binom{4}{2}\binom{3}{0}}{\binom{7}{2}} = \frac{6}{21} = \frac{2}{7}$$

Probability Mass Fun. (7/14)

Example3

Sample Space	y
RR	2
RB	1
BR	1
BB	0

y	0	1	2
$f(y) = P(Y = y)$	1/7	4/7	2/7

Probability Mass Fun. (8/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Note: Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school.

Probability Mass Fun. (8/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Note: Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. **Then x can only take the numbers 0, 1, and 2.**

Probability Mass Fun. (9/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{136}{190}$$

Probability Mass Fun. (10/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

$$f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190}$$

Probability Mass Fun. (11/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}$$

Probability Mass Fun. (12/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

x	0	1	2
$f(x) = P(X = x)$	136/190	51/190	3/190

Probability Mass Fun. (13/14)

Example5

There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next four bits transmitted. The possible values for X are $\{0, 1, 2, 3, 4\}$.

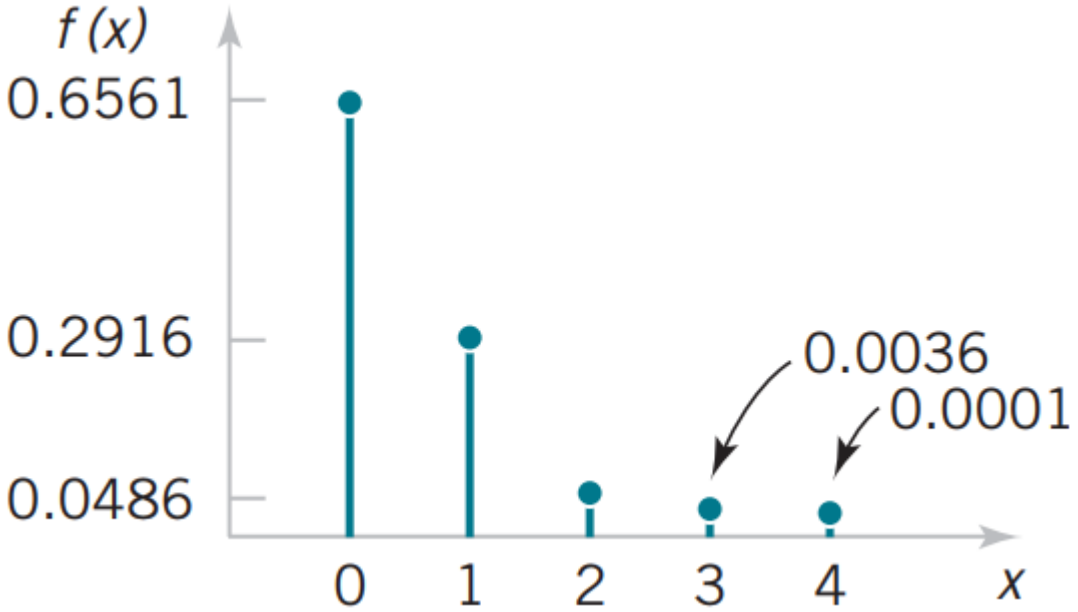
Suppose that the probabilities are

$$P(X = 0) = 0.6561 \quad P(X = 1) = 0.2916$$
$$P(X = 2) = 0.0486 \quad P(X = 3) = 0.0036$$
$$P(X = 4) = 0.0001$$

Probability Mass Fun. (14/14)

Example5

$$P(X = 0) = 0.6561 \quad P(X = 1) = 0.2916$$
$$P(X = 2) = 0.0486 \quad P(X = 3) = 0.0036$$
$$P(X = 4) = 0.0001$$



Cumulative Distribution (1/9)

The cumulative distribution function (cdf), denoted by $F(x)$, measures the probability that the random variable X assumes a value less than or equal to x, that is,

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

Cumulative Distribution (2/9)

If X is discrete, then

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Cumulative Distribution (3/9)

If X is discrete, then

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8
$\boldsymbol{F(x) = P(X \leq x)}$	**1/8**	**3/8**	**5/8**	**7/8**	**8/8**

Cumulative Distribution (4/9)

Example1

$$P(X = 0) = 0.6561 \quad P(X = 1) = 0.2916$$
$$P(X = 2) = 0.0486 \quad P(X = 3) = 0.0036$$
$$P(X = 4) = 0.0001$$

x	0	1	2	3	4
$f(x) = P(X = x)$	0.6561	0.2916	0.0486	0.0036	0.0001

Cumulative Distribution (5/9)

Example1

$$P(X=0)=0.6561 \quad P(X=1)=0.2916$$
$$P(X=2)=0.0486 \quad P(X=3)=0.0036$$
$$P(X=4)=0.0001$$

x	0	1	2	3	4
$f(x)=P(X=x)$	0.6561	0.2916	0.0486	0.0036	0.0001
$\boldsymbol{F(x)=P(X\leq x)}$	**0.6561**	**0.9477**	**0.9963**	**0.9999**	**1**

Cumulative Distribution (6/9)

Example1

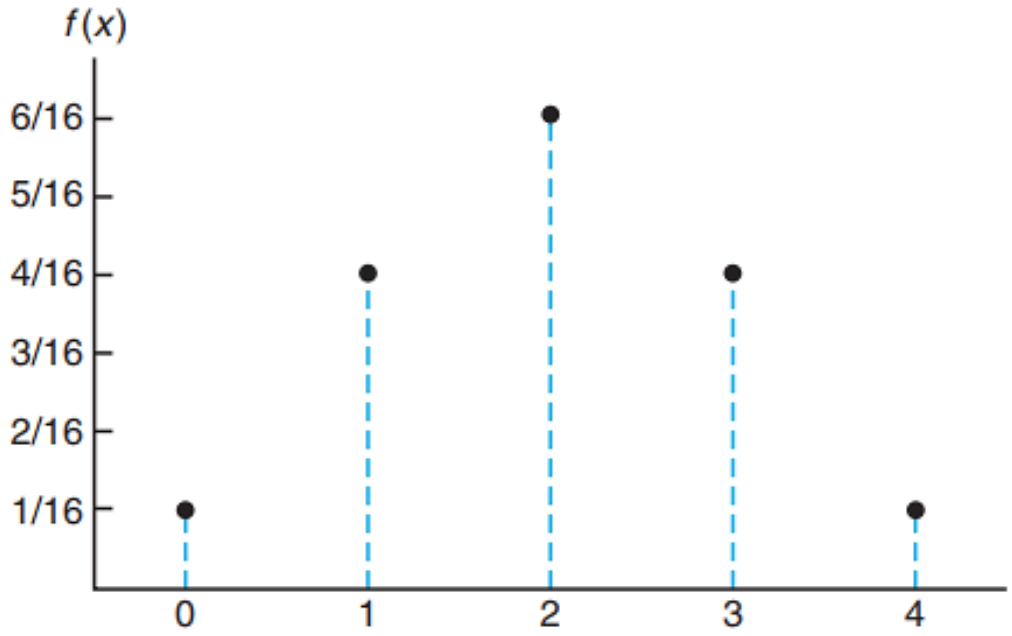
x	0	1	2	3	4
$f(x) = P(X = x)$	0.6561	0.2916	0.0486	0.0036	0.0001
$\boldsymbol{F(x) = P(X \leq x)}$	**0.6561**	**0.9477**	**0.9963**	**0.9999**	**1**

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.6561 & 0 \leq x < 1 \\ 0.9477 & 1 \leq x < 2 \\ 0.9963 & 2 \leq x < 3 \\ 0.9999 & 3 \leq x < 4 \\ 1 & 4 \leq x \end{cases}$$

Cumulative Distribution (7/9)

Example2

x	0	1	2	3	4
$f(x) = P(X = x)$	1/16	4/16	6/16	4/16	1/16



Probability mass function plot.

Cumulative Distribution (8/9)

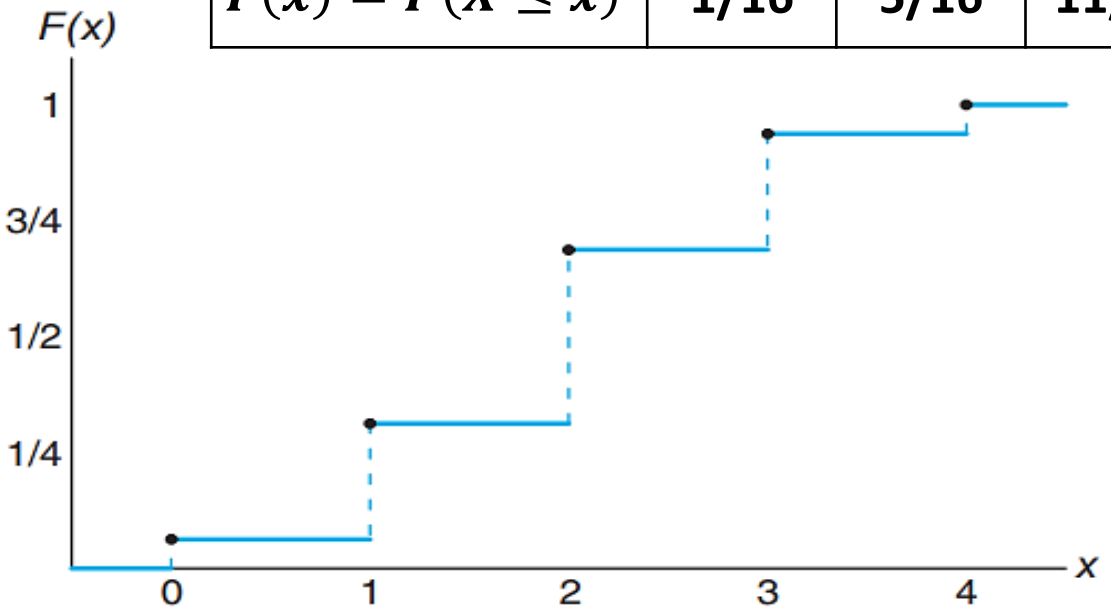
Example2

x	0	1	2	3	4
$f(x) = P(X = x)$	1/16	4/16	6/16	4/16	1/16
$\boldsymbol{F(x) = P(X \leq x)}$	**1/16**	**5/16**	**11/16**	**15/16**	**16/16**

Cumulative Distribution (9/9)

Example2

x	0	1	2	3	4
$f(x) = P(X = x)$	1/16	4/16	6/16	4/16	1/16
$\boldsymbol{F(x) = P(X \leq x)}$	**1/16**	**5/16**	**11/16**	**15/16**	**16/16**



Discrete cumulative distribution function.

Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-

Lecture #4: https://www.youtube.com/watch?v=zWDzNUTfk9s&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=4

Notes Lec 1 – 4: https://www.youtube.com/watch?v=F9f61KpLeRk&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=5

Thank You

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